On the distribution of the zeros of the Euler double zeta-function $\zeta_2(s,s)$

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The Euler double zeta-function is defined by the double series

$$\zeta_2(s,s)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-s}(m+n)^{-s}.$$

In 2014, in a joint work with M. Shōji, I studied the distribution of the zeros of $\zeta_2(s,s)$ numerically, and presented several observations. In this talk I will report that many of those observations can be proved, refined, or refuted theoretically.

(i) First, we can show that $\zeta_2(s,s)$ has no zero in the half-plane $\Re s > \sigma_b(= 1.8018642....)$.

(ii) On the other hand, if $1 < \sigma_0 < \sigma_b$ or $1/2 < \sigma_0 < 1$, then we can prove that there are infinitely many zeros in the strip $\sigma_0 - \delta < \Re s < \sigma_0 + \delta$ (where $\delta > 0$ is sufficiently small).

(iii) Under the Riemann Hypothesis (for the Riemann zeta-function $\zeta(s)$), we can find some zero-free region in the strip $0 < \Re s < 1/2$.

(iv) For $\sigma_1 < 0$, we can show that $\zeta_2(s,s)$ has no zero in the region $\sigma_1 \le \Re s < 0$ and $|\Im s|$ is sufficiently large.

Actually I will report more quantitative statements of some of the above results in the talk.

In order to prove these results, we begin with the fact that, by using the harmonic product formula

$$\zeta_2(s,s)=\frac{1}{2}(\zeta(s)^2-\zeta(2s)),$$

we may reduce the problem to the properties of $\zeta(s)$. Then, the main tools for the proof of these theorems are classical complex function theory (such as Rouché's theorem), Bohr's convex curve argument, Kronecker's approximation theorem, Voronin's universality theorem for $\zeta(s)$, and order estimates of $\zeta(s)$.

This is a joint work with Professor Tomokazu Onozuka (Oita Univ.) and Professor Isao Wakabayashi (Seikei Univ.).