Discrepancy estimates of the Riemann zeta-function with general shifts

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The study of the value distribution of zeta-functions began with Bohr's denseness theorem in the 1910s. In 1930s, Bohr and Jessen refined this result by establishing the following limit theorem. Let $\sigma > 1/2$, let R be a rectangle whose sides are parallel to the coordinate axes, and define

$$\mathbb{P}_T(\log\zeta(\sigma+it)\in R) := \frac{1}{T}\mathrm{meas}\left\{t\in[T,2T] : \log\zeta(\sigma+it)\in R\right\},$$

where meas denotes the 1-dimensional Lebesgue measure. Then the limit

$$\lim_{T\to\infty}\mathbb{P}_T(\log\zeta(\sigma+it)\in R) = W(\sigma,R)$$

exists. Subsequently, Matsumoto obtained quantitative versions of this theorem by estimating the discrepancy

$$\left|\mathbb{P}_T(\log\zeta(\sigma+it)\in R) - W(\sigma,R)\right|.$$

The best bound currently known was obtained by Lamzouri–Lester–Radziwiłł in 2019.

On the other hand, in 1975, Voronin established the universality theorem for the Riemann zeta-function. In recent years, there has been considerable interest in replacing the classical shift $+i\tau$ appearing in universality theorems by a more general shift $+i\gamma(\tau)$. More precisely, one studies whether

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[T,2T] : \sup_{s\in K}|\zeta(s+i\gamma(\tau))-f(s)|<\varepsilon\right\}>0$$

holds.

In 2024, Andersson et al. proved such a universality theorem for a broad class of shifts, including exponential shifts. Moreover, Nakai introduced a class $\mathcal{F}$ of admissible functions satisfying certain conditions and further generalized their result.

In this talk, we investigate how the discrepancy estimates change when the classical shift $+it$ is replaced by a general shift $+i\gamma(t)$. In particular, we study estimates for

$$|\mathbb{P}_T(\log\zeta(\sigma+i\gamma(t))\in R) - W(\sigma,R)|$$

for functions $\gamma\in\mathcal{F}$. Furthermore, we discuss logarithmic shifts of the form

$$\gamma(t) = (\log t)^r,$$

which are not covered by the previous works.