

Zeta Functions and Their Classes: Results and Problems in Various Aspects

~ International Workshop ~



September 10–12, 2026
Faculty of Mathematics and Informatics, Vilnius University

This workshop is organised within the framework of the bilateral Lithuania–Japan project “*From Value Distribution until Approximation: Zeta-Functions and Their Classes*” (grants S-LJP-25-1 and JPJSBP 120254202).

Organisers:

Roma Kačinskaitė (Vilnius University, Lithuania)
Kohji Matsumoto (Aichi Institute of Technology, Japan)
Benjaminas Togobickij (Vilnius University, Lithuania)

First Day, September 10 (Thur)

9:15 - 9:30 *Opening*

Chairman Athanasios Sourmelidis

9:30 - 10:20 Antanas Laurinćikas (Vilnius University)
S.M. Voronin (1946–1997) – the discoverer of universality of zeta-functions

10:30 - 11:20 Hidehiko Mishou (Tokyo Denki University)
Joint universality for shifts of the Riemann zeta function with integer multiplicities

11:30 - 12:20 Keita Nakai (Nagoya University)
Discrepancy estimates of the Riemann zeta-function with general shifts

12:30 - 12:50 Lukas Kuzma (Vilnius University)
Limit theorems for combinatorial numbers associated with Dirichlet L -functions and their application to the computation of Dirichlet L -functions values

13:00 - 14:30 *Lunch*

Chairman Ramūnas Garunkštis

14:30 - 15:20 Sumaia Saad Eddin (RISC, Johannes Kepler University Linz)
The Kummer ratio

15:30 - 16:20 Shingo Sugiyama (Kanazawa University)
Weighted one-level density for Dirichlet L -functions and automorphic L -functions

16:30 - 17:20 Olivier Ramaré (CNRS, Aix-Marseille Université)
Almost periodic character of some error terms level sets in prime number theory

Second Day, September 11 (Fri)

Chairman Igoris Belovas

9:30 - 10:20 Kohji Matsumoto (Aichi Institute of Technology)
On the distribution of the zeros of the Euler double zeta-function $\zeta_2(s, s)$

10:30 - 11:20 Kenta Endo (National Institute of Technology, Suzuka College)
Denseness of the values of Euler–Zagier type multiple zeta-functions with identical arguments on the critical line

11:30 - 12:20 Athanasios Sourmelidis (CNRS, Université de Lille)
Joint extreme values of L -functions

12:30 - 12:50 Benjaminas Togobickij (Vilnius University)
Some results on universality for the Selberg–Steuding class

13:00 - 14:30 *Lunch*

Chairman Hidehiko Mishou

14:30 - 15:20 Jörn Steuding (University of Wuerzburg)
Riemann’s functional equation, Epstein zeta-functions, and Speiser’s correspondence

15:30 - 16:20 Igoris Belovas (Vilnius University)
Studies on zeta functions: inequalities, limit theorems and problems

16:30 - 16:50 Yuichiro Toma (Waseda University)
On the extreme values of Euler-Kronecker constants

18:00 - 21:00 *Dinner*

Third Day, September 12 (Sat)

Chairman Kohji Matsumoto

9:30 - 10:20 Łukasz Páńkowski (Adam Mickiewicz University)
On mixed joint universality for general Dirichlet series

10:30 - 11:20 Hirofumi Nagoshi (Gunma University)
Universality of the Riemann zeta-function, Mertens' theorem and the prime number theorem

11:30 - 12:20 Ramūnas Garunkštis (Vilnius University)
Zeros of the derivative of the Selberg zeta function

12:30 - 12:50 *Closing*

14:00 - 16:00 *Excursion*

Studies on zeta functions: inequalities, limit theorems and problems

IGORIS BELOVAS

Vilnius University (Lithuania)

This talk surveys recent results obtained by the speaker in analytic and computational number theory. Particular attention is given to inequalities concerning absolute values of zeta functions at places symmetric with respect to the critical line, including the Riemann zeta function, Dirichlet L -functions, the modified Selberg zeta function, and the Selberg zeta function associated with compact Riemann surfaces. These inequalities are of special interest because of their close connections with the Riemann Hypothesis and its generalizations. Further results address the prime zeta function, one of the most under-researched members of the family. New results on its zero-free region are presented, together with conjectures on the distribution of its zeros. In addition, limit theorems for combinatorial numbers associated with zeta functions are examined, along with applications of these asymptotic results to the development of efficient algorithms for the numerical evaluation of zeta function values.

Denseness of the values of Euler–Zagier type multiple zeta-functions with identical arguments on the critical line

KENTA ENDO

National Institute of Technology, Suzuka College (Japan)

In this study, we consider the value distribution of Euler–Zagier type multiple zeta-functions with identical arguments. In the one-fold case, this function coincides with the Riemann zeta-function. Various results have been obtained concerning the denseness of the values on vertical lines. However, the denseness of the values of the Riemann zeta-function on the critical line remains an open problem.

On the other hand, for Euler–Zagier type multiple zeta-functions with identical arguments, one can develop arguments analogous to those for the Riemann zeta-function concerning the denseness of the values on vertical lines. However, the corresponding denseness problem on the critical line had remained unresolved for multiple zeta-functions. Recently, Mishou proved that, assuming the Riemann Hypothesis, the values of the double zeta-function on the critical line are dense in the complex plane.

In this talk, under the assumption of the Riemann Hypothesis, we show that, for Euler–Zagier type r -fold multiple zeta-functions with identical arguments and $r \geq 4$, the values on the critical line are also dense in the complex plane.

Zeros of the derivative of the Selberg zeta function

RAMŪNAS GARUNKŠTIS
Vilnius University (Lithuania)

We will provide an overview of the latest results on the zero distribution of the derivatives of Selberg zeta functions.

**Limit theorems for combinatorial numbers associated with Dirichlet L -functions
and their application to the computation of Dirichlet L -functions values**

LUKAS KUZMA

Vilnius University (Lithuania)

Modifications of algorithms for computing Dirichlet L -functions proposed by Šleževičienė are presented. These modifications rely on asymptotic approximations derived from limit theorems for the coefficients of the corresponding series. The computational complexity and performance of implementations based on Chebyshev, Hermite, Legendre, and some other polynomial families are analyzed. The proposed algorithms are further applied to the visualization of fractal structures associated with Dirichlet L -functions.

S.M. Voronin (1946–1997) – the discoverer of universality of zeta-functions

ANTANAS LAURINČIKAS

Vilnius University (Lithuania)

The first part of the talk is devoted to a short biography and scientific activity of S.M. Voronin.
In the second part, we will discuss universality of the Hurwitz zeta-function in short intervals.

On the distribution of the zeros of the Euler double zeta-function $\zeta_2(s, s)$

KOHI MATSUMOTO

Aichi Institute of Technology (Japan)

The Euler double zeta-function is defined by the double series

$$\zeta_2(s, s) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-s} (m+n)^{-s}.$$

In 2014, in a joint work with M. Shōji, I studied the distribution of the zeros of $\zeta_2(s, s)$ numerically, and presented several observations. In this talk I will report that many of those observations can be proved, refined, or refuted theoretically.

- (i) First, we can show that $\zeta_2(s, s)$ has no zero in the half-plane $\Re s > \sigma_b (= 1.8018642\dots)$.
- (ii) On the other hand, if $1 < \sigma_0 < \sigma_b$ or $1/2 < \sigma_0 < 1$, then we can prove that there are infinitely many zeros in the strip $\sigma_0 - \delta < \Re s < \sigma_0 + \delta$ (where $\delta > 0$ is sufficiently small).
- (iii) Under the Riemann Hypothesis (for the Riemann zeta-function $\zeta(s)$), we can find some zero-free region in the strip $0 < \Re s < 1/2$.
- (iv) For $\sigma_1 < 0$, we can show that $\zeta_2(s, s)$ has no zero in the region $\sigma_1 \leq \Re s < 0$ and $|\Im s|$ is sufficiently large.

Actually I will report more quantitative statements of some of the above results in the talk.

In order to prove these results, we begin with the fact that, by using the harmonic product formula

$$\zeta_2(s, s) = \frac{1}{2}(\zeta(s)^2 - \zeta(2s)),$$

we may reduce the problem to the properties of $\zeta(s)$. Then, the main tools for the proof of these theorems are classical complex function theory (such as Rouché's theorem), Bohr's convex curve argument, Kronecker's approximation theorem, Voronin's universality theorem for $\zeta(s)$, and order estimates of $\zeta(s)$.

This is a joint work with Professor Tomokazu Onozuka (Oita Univ.) and Professor Isao Wakabayashi (Seikei Univ.).

Joint universality for shifts of the Riemann zeta function with integer multiplicities

HIDEHIKO MISHOU

Tokyo Denki University (Japan)

In 2016, Pańkowski proved that the joint universality property holds for shifts of the Riemann zeta-function with distinct positive integer multiplicities d_1 and d_2 . Namely, for any compact subsets K_j of the strip $\{s \in \mathbb{C} \mid 1/2 < \Re(s) < 1\}$ with connected complements, any non-vanishing analytic functions $f_j(s)$, and any $\varepsilon > 0$, the set of real numbers τ satisfying $|\zeta(s + id_j\tau) - f_j(s)| < \varepsilon$ uniformly in $j = 1, 2$ and $s \in K_j$ has a positive lower density.

In this talk, we will extend this joint universality result to an arbitrary number of integers under certain conditions.

Universality of the Riemann zeta-function, Mertens' theorem and the prime number theorem

HIROFUMI NAGOSHI

Gunma University (Japan)

In the proof of his universality theorem for the Riemann zeta-function $\zeta(s)$, Voronin used a theorem of Markov and the prime number theorem with a certain error term. Later, Bagchi instead used a theorem of Bernstein and the prime number theorem of the form $\pi(x) = x(\log x)^{-1} + x(\log x)^{-2} + O(x(\log x)^{-3})$. In this talk, we will prove the modern form of the universality theorem for $\zeta(s)$ by modifying Bagchi's proof and using the prime number theorem (without an error term) $\pi(x) \sim x(\log x)^{-1}$, which is equivalent to $\sum_{p \leq x} p^{-1} \log p = \log x + c_0 + o(1)$. We will also prove Voronin's original universality theorem by using Mertens' theorem $\left| \sum_{p \leq x} p^{-1} \log p - \log x \right| < 2$, which is weaker than the prime number theorem.

Discrepancy estimates of the Riemann zeta-function with general shifts

KEITA NAKAI

Nagoya University (Japan)

The study of the value distribution of zeta-functions began with Bohr's denseness theorem in the 1910s. In 1930s, Bohr and Jessen refined this result by establishing the following limit theorem. Let $\sigma > 1/2$, let R be a rectangle whose sides are parallel to the coordinate axes, and define

$$\mathbb{P}_T(\log \zeta(\sigma + it) \in R) := \frac{1}{T} \text{meas} \{t \in [T, 2T] : \log \zeta(\sigma + it) \in R\},$$

where meas denotes the 1-dimensional Lebesgue measure. Then the limit

$$\lim_{T \rightarrow \infty} \mathbb{P}_T(\log \zeta(\sigma + it) \in R) = W(\sigma, R)$$

exists. Subsequently, Matsumoto obtained quantitative versions of this theorem by estimating the discrepancy

$$|\mathbb{P}_T(\log \zeta(\sigma + it) \in R) - W(\sigma, R)|.$$

The best bound currently known was obtained by Lamzouri–Lester–Radziwiłł in 2019.

On the other hand, in 1975, Voronin established the universality theorem for the Riemann zeta-function. In recent years, there has been considerable interest in replacing the classical shift $+i\tau$ appearing in universality theorems by a more general shift $+i\gamma(\tau)$. More precisely, one studies whether

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [T, 2T] : \sup_{s \in K} |\zeta(s + i\gamma(\tau)) - f(s)| < \varepsilon \right\} > 0$$

holds.

In 2024, Andersson et al. proved such a universality theorem for a broad class of shifts, including exponential shifts. Moreover, Nakai introduced a class $\mathcal{F}$ of admissible functions satisfying certain conditions and further generalized their result.

In this talk, we investigate how the discrepancy estimates change when the classical shift $+it$ is replaced by a general shift $+i\gamma(t)$. In particular, we study estimates for

$$|\mathbb{P}_T(\log \zeta(\sigma + i\gamma(t)) \in R) - W(\sigma, R)|$$

for functions $\gamma \in \mathcal{F}$. Furthermore, we discuss logarithmic shifts of the form

$$\gamma(t) = (\log t)^r,$$

which are not covered by the previous works.

On mixed joint universality for general Dirichlet series

ŁUKASZ PÁNKOWSKI

Adam Mickiewicz University (Poland)

This talk will be devoted to joint universality for general Dirichlet series, with particular emphasis on mixed joint universality phenomena involving L -functions from the Selberg class and zeta-functions of Hurwitz and Lerch type. We shall formulate general analytic and arithmetic axioms which guarantee that the corresponding L - and zeta-functions satisfy a joint universality theorem. The result can be regarded as a general framework applicable to many situations in which one aims to prove mixed joint universality.

Almost periodic character of some error terms level sets in prime number theory

OLIVIER RAMARÉ

CNRS, Aix-Marseille Université (France)

A main player in this talk will be the function defined for positive t by $\phi(t) = (\psi(e^t) - e^t)/e^{-t/2}$ where $\psi(x)$ is the summatory function of the van Mangoldt function. A proper definition of $\phi(t)$ for negative t will be provided.

Our subject of interest are the sets $E(y) = \{t \in \mathbb{R}, \phi(t) > y\}$, under the generalized Riemann Hypothesis. More than 20 years ago we proved with J. Kaczorowski that such sets have some strong almost periodic character, a study completed in an unpublished note of J.C. Schlage-Puchta where it is shown that indeed these sets (or more precisely their characteristic functions) are almost periodic in the L^2 -Stepanov sense (except for y in some at most countable subset of real numbers). The first object of this talk is to set this background and to recall these results.

The second object of this talk is to detail some open questions. A major one is this: what are the periods of the set $E(y)$? We showed that they lie in the $\mathbb{Q}$ -vector space generated by the ordinates of the zeroes of the Riemann zeta-function, a poor man's conjecture is that these ordinates should exactly be the periods but the numerical knowledge is missing to confirm such a conjecture.

This works readily extends to the Selberg class.

The Kummer ratio

SUMAIA SAAD EDDIN

RISC, Johannes Kepler University Linz (Austria)

Let q be a prime and consider the cyclotomic field $\mathbb{Q}(\zeta_q)$. The ratio of its class number to the class number of its maximal real subfield $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$ is known as the *relative class number* and is denoted by $h_1(q)$.

In 1851, Kummer claimed that $h_1(q)$ is asymptotically equal to an explicit elementary function $G(q)$. The quotient

$$R(q) := \frac{h_1(q)}{G(q)}$$

is now known as the *Kummer ratio*. Understanding the extent to which $R(q)$ deviates from 1 has proved to be a subtle problem, closely connected with deep questions concerning the distribution of prime numbers.

In this talk, I will survey recent advances on the irregular behavior of the Kummer ratio and discuss related phenomena arising in the study of Euler–Kronecker constants associated with the fields $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_q + \zeta_q^{-1})$.

The talk is based on joint work with Neelam Kandhil, Alessandro Languasco, Pieter Moree and Alisa Sedunova.

Joint extreme values of L -functions

ATHANASIOS SOURMELIDIS

CNRS, Université de Lille (France)

In this talk we will discuss about the extremal behavior of L -functions. As a special case we will see that any tuple of distinct primitive $\mathrm{GL}(1)$ and $\mathrm{GL}(2)$ L -functions can simultaneously attain large values on the critical line. This is an unconditional improvement of a general result due to Heap and Li who have assumed the Riemann Hypothesis for more than three such L -functions. The main machinery is the resonance method of Soundararajan and a variation of Heath-Brown's method for the fractional moments of the Riemann zeta-function. The latter method, in particular, builds on Gabriel's convexity theorems and makes it possible to avoid using any information on the zero distribution of L -functions whose degree is less than three.

Riemann’s functional equation, Epstein zeta-functions, and Speiser’s correspondence

JÖRN STEUDING

University of Wuerzburg (Germany)

Two hundred years ago Riemann was born. In his path-breaking paper from 1859 he proved the functional equation of the (Riemann) zeta-function and his several unproven claims led to a better understanding of the distribution of prime numbers. Only his famous “Riemann Hypothesis” (that all complex zeros of his zeta-function lie on the critical line) is not yet solved. In the talk we consider the so-called Epstein zeta-functions, satisfying a certain Riemann-type functional equation, and discuss their zero-distributions. For some of these zeta-functions a variation of “Riemann Hypothesis” is expected to hold, for others definitely not (for which we will give various explicit examples). We sketch a proof of a variant of Speiser’s celebrated correspondence for Epstein zeta-functions (and their derivatives). Finally, we touch questions about the distribution of values of these zeta functions and “universality”.

This is joint work with my Rasa Steuding.

Weighted one-level density for Dirichlet L -functions and automorphic L -functions

SHINGO SUGIYAMA

Kanazawa University (Japan)

At the end of the 20th century, a relation between number theory and physics was considered by Katz-Sarnak. Katz and Sarnak suggested that zeros of L -functions should distribute like eigenvalues of random matrices. In this talk, we suggest the weighted density conjecture of zeros for families of L -functions. Moreover we show several pieces of evidence supporting this conjecture, such as symmetric power L -functions attached to modular forms, Dirichlet L -functions, shifted Riemann zeta functions and quadratic Dirichlet L -functions. The results on Dirichlet L -functions and on shifted Riemann zeta functions are joint work with Ade Irma Suriajaya (Kyushu University). The result on quadratic Dirichlet L -functions is joint work with Kohta Gejima (Otani University).

Some results on universality for the Selberg–Steuding class

BENJAMINAS TOGOBICKIJ

Vilnius University (Lithuania)

Let $\mathcal{S}$ be the Selberg class, i.e. the class of Dirichlet series satisfying the Ramanujan hypothesis, having a meromorphic continuation with at most a pole at $s = 1$, a functional equation of Riemann type, and an Euler product. Then the Selberg–Steuding class $\mathcal{S}$ consists of those L -functions from the class $\mathcal{S}$ that additionally satisfy the prime mean-square condition of the form

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a(p)|^2 = \kappa > 0,$$

a kind of prime number theorem for the Dirichlet coefficients.

In 1975, Voronin proved that the shifts of the Riemann zeta-function $\zeta(s + i\tau)$ approximate uniformly on compact subsets of the right half-plane of the critical strip any non-vanishing analytic function. In this talk, we discuss some universality results for the Selberg–Steuding class $\mathcal{S}$.

On the extreme values of Euler-Kronecker constants

YUICHIRO TOMA

Waseda University (Japan)

The Euler-Kronecker constant, introduced by Yasutaka Ihara in the 2000s, is a natural generalization of the classical Euler-Mascheroni constant to number fields. The Euler-Mascheroni constant plays a fundamental role in analytic number theory and appears in many asymptotic formulas involving prime numbers and harmonic sums.

Despite considerable progress in recent years, many aspects of the value distribution of Euler-Kronecker constants remain mysterious, even in the case of Abelian extensions. Since the Euler-Kronecker constant can be expressed as the logarithmic derivative of the Dedekind zeta function at $s = 1$, its value distribution is closely related to the possible existence of Siegel zeros. In this talk, we report an result on extreme value estimates of Euler-Kronecker constants of some abelian fields under the Generalized Riemann Hypothesis.